Odd Perfect Numbers

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1 Introduction

1.1 What are Perfect Numbers?

In math, there are what's called *perfect numbers*. If the number n is perfect, the sum of its divisors equal $2n$. This is denoted $\sigma(n) = 2n$, where σ (sigma) is the divisor function.

The number 28, for example, is a perfect number, because

$$\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 2 \cdot 28$$

.

Currently, all known perfect numbers are even and it has yet to be proven that odd perfect numbers exist or not. This document will explore the concept of odd perfect numbers and whether or not one could possibly exist.

1.2 Status of the Problem

As of today, only 51 known perfect numbers have been found (all even). The last one, found 8 April 2018, has nearly 50 million digits, so it's clear a brute force tactic isn't going to work. With exhaustive searches, however, it is known that there is no odd perfect number below $10^{1500} \sim 2^{4983}$. A variety of restrictions on what an odd perfect number could look like are also known, but none of them have so far been able to disprove their existence. In this document, we reveal different theorems and lemmas that might be helpful in the proof that either odd perfect numbers don't exist, or that they do.

1.3 What to expect in this document

The first thing we explore in this document are triperfect numbers. We discuss their relation to perfect numbers as well as why they are an important step in solving the odd perfect numbers question. Besides triperfects, another approach we took utilizing modular arithmetic. Should odd perfects exist, we found a couple modular restrictions on them and their prime factors that may prove useful in later on. We also devised an approach called the multiplying method. Essentially, it helps describe the relationship between n and $\sigma(n)$, as well as their ratio to each other.

2 A Deeper Look into Perfect Numbers

2.1 What is a triperfect number?

We've already described what a perfect number is, but what about triperfect numbers? Similar to a perfect number, a *triperfect* number is a number whose divisors sum to three times the number's value. So, $\sigma(n) = 3n$. The significance of a triperfect number comes from the fact that an odd perfect number multiplied by 2 will equal a triperfect number, if odd perfect numbers exist of course. We will expand upon this later in the document.

2.2 Perfect Numbers and Mersenne primes

The majority of perfect numbers have been found using their relation to Mersenne primes. To understand why, lets first discuss what a Mersenne prime is. A Mersenne prime is a prime number which takes the form $2^n - 1$; for example, 3, 7, or 31. The theorem, first proven by Euclid around 300 B.C.E., is that if $2^n - 1$ is a Mersenne prime, then $2^{n-1}(2^n - 1)$ must be perfect number. All perfect numbers are of this form, and each Mersenne prime has it's own corresponding perfect number, but not necessarily the other way around, should an odd perfect number exist. We show this here in the proof below, and expand upon it in section 3.4. Before we do, we'll need two lemmas.

Lemma 1. *If $p = 2^k - 1$, then $\sigma(2^{k-1}) = p$.*

Lemma 2. *If $p = 2^k - 1$, where p is a Mersenne prime, then $\sigma(p) = 2^k$.*

Theorem 1. *All even perfect numbers are the product of a power of 2 and a Mersenne prime.*

Proof. Assume p, is a prime number of the form $2^k - 1$, and $n = (2^k - 1) \cdot 2^{k-1}$.

$$\sigma(n) = \sigma(p) \cdot \sigma(2^{k-1}) = 2^k(2^k - 1)$$

$$\sigma(n) = 2^k(2^k - 1) = 2(2^k - 1)(2^{k-1}) = 2n$$

Therefore, since $\sigma(n) = 2n$, n must be a perfect number if it is of the form, $2^{n-1}(2^n - 1)$. □

2.3 General Equations

A couple equations and relationships between n and $\sigma(n)$ occur often in this paper, so they will be proven once here and from here on only referenced.

Lemma 3. *Given two numbers a and b, where a and b are coprime, then*

$$\sigma(ab) = \sigma(a) \cdot \sigma(b)$$

which not only works for two coprime factors, but an infinite amount.

Proof. a and b have no relationship to each other in their factors. Because of this every factor of ab will simply be a factor of a and of b multiplied together. These can be taken apart, and remultiplied as the original $\sigma(n)$ function of both a and b. ☐

Lemma 4.

$$\sigma(p^k) = \Sigma_{i=0}^{k} p^i$$

where p is any prime number.

Proof. Because p is a prime, the only numbers which can divide p are powers of p. Each power of p divides p in the relationship $p^k = p^i \cdot p^{k-i}$. Therefore the sum of all divisors of p^k will be the sum of all powers of p below and including k. ☐

Any n can be rewritten as the product of all of it's prime factors, represented by the set

$$p_1^{k_1}, p_2^{k_2}, ..., p_m^{k_m}$$

Which by the first equation described can be rewritten as

$$\sigma(p_1^{k_1}) \cdot \sigma(p_2^{k_2}) \cdot \sigma(p_3^{k_3})...$$

By the second relationship, this is then equivalent to

$$(p_1^0 + p_1^1 + p_1^2 + ...p_1^{k_1}) \cdot (p_2^0 + p_2^1 + p_2^2 + ...p_2^{k_2}) \cdot (p_3^0 + p_3^1 + p_3^2 + ...p_3^{k_3})...$$

This relationship can be justified by the visual described below, and will be used repeatedly though this paper.

2.4 An alternate way of visualizing Perfect Numbers

An alternate way to visualize a perfect number would be to take the prime factors of a number n to be the set

$$[p_0^{k_0}, p_1^{k_1}, ..., p_{m-1}^{k_{m-1}}, p_m^{k_m}].$$

The sum of the divisors of this number can be visualized in terms of the prime divisors. A single prime divisor p_i raised to some power k_i can be seen as a line of numbers length $k_i + 1$, where each number is p_1 raised to an increasing power, from 0 to k_i. A second prime divisor can be represented as a new dimension, orthagonal to the previous dimension, with the same properties.

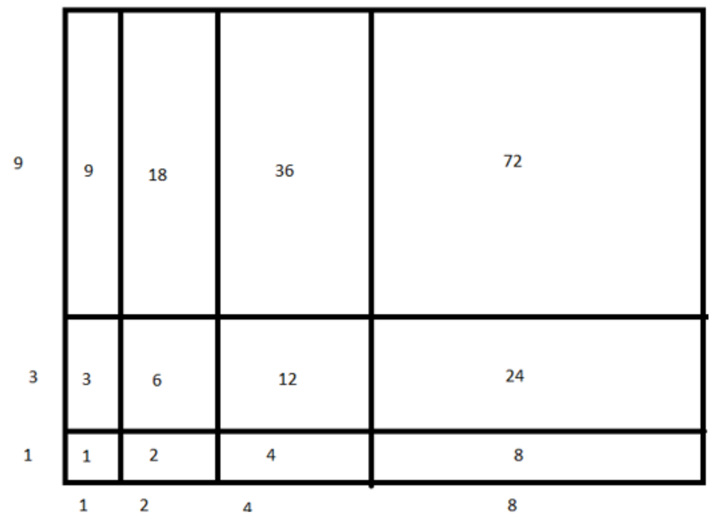

Figure 1: A visualization of factors of 72

The axes of these new dimensions would be logarithmic in scale with different bases. The area between these two axes can be filled with their multiples, so that the position (i, j) would be represented by $p_0^i \cdot p_1^j$. Using this method, the number of dimensions will be equal to the number of prime factors, which all together represents all of the factors of n.

Example 1. Take $n = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11$.

Using the above method, the first axis will be represented by the values $1, 2, 4, 8$. The second axis will be $1, 3, 9$, and the area between them will hold all possible combinations of the two, namely $6, 18, 12, 36, 24, 72$. Then the third and fourth axes can be filled in as $1, 5, 25$ and $1, 11$, respectively. In the end you will get a four dimensional hypercube with side lengths $3, 2, 2, 1$.

3 Properties of Perfect Numbers

3.1 Perfect Numbers in relation to Triperfect Numbers

As we've already explained, a number n is triperfect iff $\sigma(n) = 3n$. This could be useful in proving the existence of odd perfect numbers, because odd perfect numbers times 2 equal a triperfect number, however, not all triperfect numbers divided by two yield an odd perfect number. So for any odd perfect number that we find, we just need to divide by two and check if the outcome is an odd perfect number. As of August 2019, 6 triperfect numbers have been found. So, if the list of triperfects is complete, then the odd perfect numbers question will have been solved such that there exists no odd perfect numbers. It's for this reason the triperfect numbers play a significant role in answering the odd perfect numbers question.

Theorem 2. *Odd perfect numbers are half of a triperfect number.*

Proof. An odd perfect number n will be coprime with 2, meaning that when n is doubled, the equation

$$\sigma(2n) = \sigma(n) \cdot \sigma(2)$$

will hold true by the equations in part 2.3. Using those equations again,

$$\sigma(2n) = \sigma(n) \cdot (1 + 2)$$

Meaning that the value of $\frac{\sigma(2n)}{2n}$ is increased by a factor of $\frac{3}{2}$ from it's original value. Since it was originally a perfect number, that now means it will be a triperfect number. □

3.2 Modular Restrictions on Odd Perfect Numbers

Theorem 3. *Any number n which is of the form* $6k - 1$*, cannot be perfect.*

Proof. First, when looking at $n = 6k - 1$, it's clear that n is congruent to $-1 \pmod 3$ or $2 \pmod 3$. Next, we will define all prime numbers that divide n as p^k. Due to the nature of prime numbers, all p^k will either be $1 \pmod 3$ or $2 \pmod 3$. As such, there must be at least one p, raised to the power of k, which is $2 \pmod 3$, so that n can also be $2 \pmod 3$. If we look at this p and k, we can reason that the power k must be an odd number, because any number congruent to $2 \pmod 3$ raised to an even power will result in $1 \pmod 3$, which violates the conditions of p^k. As a result, k can be represented as the set of odd numbers, $2j + 1$.

Proposition 1: $\sigma(p^k) = \sum_{i=0}^{k} p^i \equiv 0 \pmod 3$
Proof by mathematical induction: Base case, $j = 0$,

$$\sigma(p^{2(0)+1}) \equiv 0 \pmod 3$$

$$\sigma(p) \equiv 0 \pmod 3$$

This is because since p is prime and already congruent to 2 (mod 3), its only factors are 1 and itself, which sum up to $p + 1$, or 0 (mod 3).

Induction Step:

$$\sigma(p^{2(j+1)+1}) \equiv 0 \pmod 3$$

$$\sigma(p^{2j+1}) + p^{2j+2} + p^{2j+3} \equiv 0 \pmod 3$$

$$\sigma(p^{2j+1}) + p^{2j+2} + p(p^{2j+2}) \equiv 0 \pmod 3$$

$$\sigma(p^{2j+1}) + p^{2j+2}(p+1) \equiv 0 \pmod 3$$

$$0 \pmod 3 + 0 \pmod 3 \equiv 0 \pmod 3$$

$$0 \pmod 3 \equiv 0 \pmod 3$$

Finally, knowing that $\sigma(p^k)$, is congruent to $0 \pmod 3$, we know that the sum of the factors of n, $\sigma(n)$, is also congruent to $0 \pmod 3$. For n to be perfect it must satisfy $2n = \sigma(n)$. However,

$$2n = 2(6k-1) \equiv 1 \pmod 3 \neq 0 \pmod 3$$

Therefore any number n, of the form $6k-1$, cannot be perfect.

□

Although the above proof is a nice endeavour of this modular approach, it can be generalized to many different cases. In the general case it is important to look how the modularity of the sum $1 + p + p^2 + ... + p^{k-1} + p^k$ changes as k increases. From the hypercube understanding, this will determine the modularity of a whole dimension, which will take effect when multiplied by every other leg.

Claim 1. *A number n in the form 1 or 2 mod 3 cannot be perfect if it has a factor of p^k where $p \equiv 1$ mod 3 and $k \equiv 2$ mod 3*

Proof. Any factor of p^k with $p \equiv 1$ mod 3 and $k \equiv 2$ mod 3 will have a factors of

$$1, p, p^2 ...$$

with modularity 3 residues

$$1, 1, 1, ...$$

For which the mod 3 version of the summation will be

$$1, 2, 0, ...$$

meaning that whenever $k \equiv 2$ mod 3, the summation will be in the form 0 mod 3. This would mean that $\sigma(n)$ has a factor of 3. For n to be perfect it would also have to have a factor of 3, which is impossible by the constraints of the claim. □

Methods similar to this can be used for many modularities, and will continue to be used. It can be used as a powerful method to restrict possible values and better understand equations stemming from other inquiries, and could be a key component in an eventual proof of the question.

3.3 The Multiplying Method

When some number is multiplied by a prime, $\sigma(n)$ changes in ways useful to understanding the problem.

Lemma 5. *Take some number n with $\sigma(n)$ and a prime p which is relatively prime to n.*

$$\sigma(n \cdot p^k) = \sigma(n) \cdot \sum_{i=0}^{k} p^i$$

Proof. This is justified by an expansion of the equations in "2.3 General Equations". □

Corollary 1.

$$\frac{\sigma(n \cdot p^k)}{n \cdot p^k} = \frac{\sigma(n)}{n} \cdot \sum_{i=0}^{k} p^{i-k}$$

Lemma 6.

$$S = \sum_{i=0}^{k} a^i = \frac{a^{k+1} - 1}{a - 1}$$

Proof.

$$S = 1 + a + a^2 + ... + a^{k-1} + a^k$$
$$S = 1 + a(a + a^2 + ... + a^{k-2} + a^{k-1})$$
$$S = 1 + a(a + a^2 + ... + a^{k-2} + a^{k-1} + a^k) - a^{k+1}$$
$$S = 1 + a \cdot S - a^{k+1}$$
$$S(1 - a) = 1 - a^{k+1}$$
$$S = \frac{a^{k+1} - 1}{a - 1}$$

□

Corollary 2.

$$\lim_{k \to \infty} \frac{\sum_{i=0}^{k} a^i}{a^k} = \frac{a}{a - 1}$$

This method can be used to modify $\frac{\sigma(n)}{n}$ by multiplying by primes which aren't prime factors of n, raised to a power. It can also be determined how far the value of $\frac{\sigma(n)}{n}$ can be raised, using the equation for the infinite sum above.

Example 2. Modify n = 31 to a perfect number by multiplying by a power of 2.

$$\frac{\sigma(n)}{n} = \frac{32}{31}$$

$$\frac{32}{31} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) = \frac{32}{31} \cdot \frac{31}{16} = 2$$

if $\frac{\sigma(m)}{m} = 2$ then m is perfect, therefore $31 \cdot 16$ is perfect

Example 3. Can n = 35 be modified to have a $\frac{\sigma(n)}{n}$ ratio of larger than 2 when multiplied by a power of 3?

$$\frac{\sigma(n)}{n} = \frac{48}{35}$$

Take the power of 3 to be arbitrarily large, which allows the use of the equation for the infinite series where a = 3.

$$\frac{48}{35} \cdot \frac{3}{3-1} = \frac{72}{35} > 2$$

Meaning that yes, when multiplied by a power of 3, 35 can have a ratio of larger than 2, and by this method could potentially become perfect.

Characteristics of n

For a number n to have a ratio $\frac{\sigma(n)}{n} = c$ when multiplied by some relatively prime a^b, what characteristics must n have?

Claim 2. *For some n and $\sigma(n)$ multiplied by some a^b to make $\frac{\sigma(n \cdot a^b)}{n \cdot a^b} = c$, the following equation must hold true*

$$\frac{\sigma(n)}{a \cdot \sigma(n) - c \cdot n \cdot (a-1)} = a^b$$

Proof.

$$\frac{\sigma(n)}{n} \cdot \frac{a^{b+1}-1}{(a-1) \cdot a^b} = \frac{\sigma(n \cdot a^b)}{n \cdot a^b} = c$$

$$a - \frac{1}{a^b} = c \cdot \frac{n}{\sigma(n)} \cdot (a-1)$$

$$\frac{\sigma(n)}{a \cdot \sigma(n) - c \cdot n \cdot (a-1)} = a^b$$

□

Example 4. Why do even perfect numbers occur when a Mersenne prime is multiplied by a power of 2?

Take n as a Mersenne prime, meaning that $\sigma(n) = n + 1$, since n is a prime. a will be 2, to multiply by a power of 2, and c will be 2, to be perfect.

$$\frac{n+1}{2(n+1) - 2n(2-1)} = 2^b$$

$$\frac{n+1}{2n+2-2n} = 2^b$$

$$n + 1 = 2^{b+1}$$

which will hold true for Mersenne primes, since they take the form $2^x - 1$

Lower Bounds on n

Question 1. What is the smallest value n can take given certain conditions such that n could potentially form a perfect number when multiplied repeatedly by a prime?

Take the equation determined above, and then rewrite it

$$\frac{\sigma(n)}{a \cdot \sigma(n) - c \cdot n \cdot (a-1)} = a^b$$

$$\sigma(n) = a^b \cdot (a \cdot \sigma(n) - c \cdot n \cdot (a-1))$$

Now take the set of numbers represented by $\sigma(n) = i \cdot n + k$. These are the set of numbers $m \cdot p = n$, which has factors of $n, \frac{n}{f_1}, \frac{n}{f_2} ..., f_2, f_1, 1$, and will keep these factors no matter the size of p, and by continuation the size of n. One example is $n = 35p$, with factors $n, \frac{n}{5}, \frac{n}{7}, \frac{n}{35}, 35, 7, 5, 1$, such that $\sigma(n) = \frac{48}{35}n + 48$.

$$in + k = a^b(ain + ak - cna + cn)$$

$$n(i - ia^{b+1} + ca - c) = a^{b+1}k - k$$

$$n = \frac{k(a^{b+1} - 1)}{i(1 - a^{b+1}) + c(a-1)}$$

The lowest this can be is where $b = 0$, leaving

$$n \geq \frac{k(a-1)}{i(1-a) + c(a-1)} = \frac{k}{c-i}$$

Which leaves a shaky lower bound.

Example 5. Take $n = 35 \cdot p$ and $\sigma(n) = \frac{48}{35}n + 48$, which was shown above. Using the equation above, n has a lower bound of

$$n \geq \frac{48}{2 - \frac{48}{35}} = \frac{48 \cdot 35}{22}$$

Which means that the prime p in question has to be larger than $\frac{48}{22}$.

However, since these are the lower bounds, they will not take effect until we reach very large numbers. A computational lower bound has already been shown of around 10^{1500}, meaning that n will have to be something incomprehensibly big according to this equation to show anything useful.

Arbitrarily large $\frac{\sigma(n)}{n}$

Claim 3. *There is no upper bound on the value of* $\frac{\sigma(n)}{n}$.

Proof. There are two ways to prove this. The first is to start with 1, for which $\frac{\sigma(n)}{n} = 1$, and then multiply by increasingly large primes.

$$\frac{\sigma(1 \cdot 2)}{2} = 1 \cdot (1 + \frac{1}{2})$$

$$\frac{\sigma(2 \cdot 3)}{2 \cdot 3} = 1(1 + \frac{1}{2})(1 + \frac{1}{3})$$

$$\frac{\sigma(2 \cdot 3 \cdot 5)}{2 \cdot 3 \cdot 5} = 1(1 + \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{5})$$

which approaches

$$\frac{\sigma(n)}{n} = \lim_{m \to \infty} \prod_{i=0}^{m} 1 + \frac{1}{p} > \lim_{m \to \infty} \sum_{i=0}^{m} \frac{1}{p}$$

which is known to diverge to ∞, meaning that there is no upper bound on $\frac{\sigma(n)}{n}$.

The second proof is very similar, again beginning with n = 1. Multiply by increasingly large primes p raised to a power k approaching ∞. Using the equation above these will approach $\frac{p}{p-1}$.

$$\lim_{k \to \infty} \frac{\sigma(1 \cdot 2^k)}{2^k} = 1 \cdot \frac{2}{2-1} = 2$$

n is now changed to 2^∞ with $\frac{\sigma(n)}{n} = 2$.

$$\lim_{k \to \infty} \frac{\sigma(n \cdot 3^k)}{n \cdot 3^k} = 2 \cdot \frac{3}{3-1} = 2 \cdot \frac{3}{2}$$

This process is continued, each time multiplying n by the next greatest prime raised to a power approaching ∞ and the value of $\frac{\sigma(n)}{n}$ being multiplied by $\frac{p}{p-1}$. When the number of times this process is repeated approaches ∞, the value of $\frac{\sigma(n)}{n}$ looks like below.

$$\frac{\sigma(n)}{n} = \lim_{m \to \infty} \prod_{i=0}^{m} \frac{p}{p-1} > \lim_{m \to \infty} \sum_{i=0}^{m} \frac{p}{p-1} > \lim_{m \to \infty} \sum_{i=0}^{m} p$$

which has no upper bound, since the sum of all primes is known to diverge. □

Approaching any ratio $\frac{\sigma(n)}{n}$

Theorem 4. *Numbers can be created with ratio $\frac{\sigma(n)}{n}$ arbitrarily close to any number c.*

Proof. Begin with a number n such that $\frac{\sigma(n)}{n}$ is close to c. Take the primes surrounding $\frac{1}{c-\frac{\sigma(n)}{n}}$, p_1 and p_2, and multiply n by these.

$$\frac{\sigma(n \cdot p_1)}{n \cdot p_1} = \frac{\sigma(n)}{n} \cdot (1 + \frac{1}{p_1}) < \frac{\sigma(n)}{n} \cdot (1 + c - \frac{\sigma(n)}{n}) = c$$

and

$$\frac{\sigma(n \cdot p_2)}{n \cdot p_2} = \frac{\sigma(n)}{n} \cdot (1 + \frac{1}{p_2}) > \frac{\sigma(n)}{n} \cdot (1 + c - \frac{\sigma(n)}{n}) = c$$

which means that when multiplied by p_1 or p_2, the value of the new $\frac{\sigma(n)}{n}$ will bound c, and be closer on both sides than the original.

Similarly this can also be understood through integer values. Define $d = \sigma(n) - 2n$, such that for numbers below perfect d is negative. The following equations hold to show the same thing as above in effect.

$$d_0 \cdot p_0 + \sigma(n) = d_1$$

where p is the nearest prime which keeps the outcome negative or zero. It can be continued with

$$d_1 \cdot p_1 + \sigma(n) \cdot (p_0 + 1) = d_2$$

$$d_2 \cdot p_2 + \sigma(n) \cdot (p_0 + 1) \cdot (p_1 + 1) = d_3$$

and so forth. If the prime in question is the lowest possible value it can take while keeping the outcomes less than or equal to zero, the outcome will decrease from the previous step. However if that initial number is not a prime, and the first prime to be found is somewhat larger, the outcome decreases from zero.

This creates the formal set of recursive equations

$$p_m = \lceil \frac{\sigma(n) \prod_{i=0}^{m-1} (p_i + 1)}{d_m} \rceil$$

given that

$$d_{m+1} = \sigma(n) \prod_{i=0}^{m-1} (p_i + 1) + d_m \cdot p_m$$

However this process is guaranteed to never find an odd perfect number by the proofs in 3.5. □

Example 6. Take the original $n = 3^3 \cdot 5^3 = 3375$ with $\sigma(n) = 6240$. Here d is equal to $6240 - 2 \cdot 3375 = -510$, and the first equation becomes

$$-510 \cdot p_0 + 6240 = d_1$$

With this equation the prime is $\lceil \frac{\sigma(n)}{-d_0} \rceil$, the nearest number which satisfies the equation given that d_1 must still be negative. If this number is not a prime the nearest larger prime will suffice. The equation filled in then will look like this

$$p_0 = \lceil \frac{6240}{510} \rceil = 13$$

$$-510 \cdot 13 + 6240 = -390$$

And then this process can be repeated, except with a new d and a new $\sigma(n)$ as such

$$-390 \cdot p_1 + 6240(13 + 1) = d_2$$

3.4 Prime Powers of odd perfect numbers

Given some number n with a set of prime factors $p_1^{k_1}, p_2^{k_2}, ..., p_m^{k_m}$, and using the equaitons detailed in part 2.3, $\sigma(n)$ can be rewritten as

$$(p_1^0 + p_1^1 + p_1^2 + ...p_1^{k_1}) \cdot (p_2^0 + p_2^1 + p_2^2 + ...p_2^{k_2}) \cdot (p_3^0 + p_3^1 + p_3^2 + ...p_3^{k_3})...$$

This expression above is congruent to 2mod4, because it is still equal to 2m where m is our odd perfect number. p_1 is odd, so if k_1 was odd, then $(p_1^0 + p_1^1 + p_1^2 + ...p_1^{k_1})$ would be even, and vice versa. Since all p are odd, this applies to every p. From that, we can form some restrictions on the powers of the prime factors of m. One restriction is that not all the k can be odd, because then you end up with many even factors multiplying out which will never be congruent to 2mod4. This is the same for all the k being even; many odd factors multiplying together will never be congruent to 2mod4. On a side note, this also implies some restrictions on the prime construction of 2m. To achieve a product of the form $4k + 2$ (2mod4), we need one even factor while the rest can be even. This is because $4k + 2 = 2(2k + 1)$ Taking this to k, we need one k to odd (such that k is congruent to 1mod4) while the rest need to be even. So all but one of the prime powers of m needs to raised to an even power, while the other one needs to be raised to an odd power. Note that this is equivalent to a square multiplied by a single prime.

Claim 4. *The only prime factor in a potential odd perfect number raised to an odd power will be a prime $p \equiv 1 mod 4$ with power $k \equiv 1 mod 4$.*

Proof. There are two possible ways to get a factor of 2mod4

1. $p \equiv$ 1mod4 raised to k
2. $p \equiv$ 3mod4 raised to k

For the first case, the modular congruencies of $1 + p + p^2 + ... + p^k$ as k increases will take the form $1, 2, 3, 0, 1...$mod4, which repeats. In this form the only possible value of $2 mod 4$ is when k is 1mod4.

For the second case, the congruencies of the sum of p^k will be in the form $1, 0, 1, 0, ...$mod4. For this list there is no possibility for the sum to be $2 mod 4$.

□

Example 7. Take $p = 5$ and $k = 1$ or $k = 5$. For 5^1, $(1 + 5) = 6 \equiv$ 2mod4, and for 5^5, $(1 + 5 + 25 + 125 + 625 + 3125) = 3906 \equiv$ 2mod4.

These conditions also lead to the conclusion that any potential odd perfect number must come in the form 1 mod 4.

Claim 5. *Any odd perfect number must come in the form 1 mod 4.*

Proof. Any odd perfect number must come in the form $m^2 \cdot (4p+1)$, and any square can only be 0 or 1 mod 4. Given that the number must be odd for this problem, the square term can only come in terms of 1 mod 4. When it is then multiplied by a number 1 mod 4, it will stay 1 mod 4. □

4 further questions and hypothesis

1. Investigate numbers $n \cdot p^k$ where $p \equiv 1 \bmod 4$ and $k \equiv 1 \bmod 4$.

   Is there anything that can be proven about specifically this case, maybe by combining a modular understanding with the multiplying method.

2. Investigate characteristics of a square and it's divisor sum.

   Can anything concrete be proven about squares besides that their value of $\sigma(n)$ is odd? If so can this be used to exclude the possibility of perfect numbers coming in the form of a square multiplied by a prime?

3. Is there any upper bound that can be shown on any variable?

   If an upper bound could be shown it might be combined with a lower bound using a computational search and restrictions involving a modular search to prove that no solution can exist.

4. Can it be proven that there is a finite number of triperfect numbers, or for that matter any n-perfect numbers?

5. Are there other reliable ways to find even or odd n-perfect numbers?

   This would give a better understanding of the problem as a whole, as well as proving the problem. As noted if there are a finite number of triperfect numbers which can be found and tested by a computer the problem would be solved. However it might also show that the only n-perfect numbers which occur reliably are even perfect numbers in the Mersenne form discussed.

Hypothesis

Currently, even perfect numbers can be reliably found through Mersenne primes. However, it is difficult to consistently find other even n-perfect numbers. As a result of our work, we are lead to believe that odd n-perfect numbers very likely do not exist.