Work

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Work

The **work** W that a *constant* force F does when moving an object over distance d is $W = Fd$.

The unit of work is the unit of force times the unit of distance. In SI units, force is measured in newtons (N) and distance is measured in meters (m), so work is measured in newton-meters, also called **joules** (J). In the British system, work is measured in foot-pounds.

If the force is not constant, then calculating work is more complicated. Suppose the force is given by a function $F(x)$, where x is the position of the object. We want to find the work done moving an object from $x = a$ to $x = b$. So we divide the interval $[a, b]$ into n equal subintervals of width Δx, and we choose a point x_i from the i^{th} subinterval. If the subintervals are small, then the force on each subinterval is approximately the constant $F(x_i)$, so the work required to move across the i^{th} subinterval is approximately $F(x_i)\Delta x$, and the total work is approximately $\sum_{i=1}^{n} F(x_i)\Delta x$. The actual work is the limit of this sum as $n \to \infty$. Since this is a Riemann sum, the answer is:

$$W = \int_a^b F(x)\,dx$$

Springs

One place where work arises is when stretching or compressing springs. To answer questions about springs, we need to know the relationship between force and distance.

Hooke's Law: The force required to hold a stretched or compressed spring a distance of x units from its natural length is $F(x) = kx$, where $k > 0$ is a constant (called the spring constant; this depends on the nature of the spring).

Example 1

A spring with natural length $5\ m$ and spring constant $k = 30\ N/m$ is stretched $10\ m$ from its natural length. Find the work required.

length. Find the work required.

1

Solution

$$W = \int_0^{10} F(x)\,dx = \int_0^{10} 30x\,dx = 1500\ J$$

2
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integral(30*x,x,0,10)
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1500
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Example 2

A spring has a natural length of $20\ cm$. A $40\ N$ force is required to stretch and hold the spring at a length of $30\ cm$. How much work (in joules) is done in stretching the spring from $35\ cm$ to $38\ cm$?

Solution In order to calculate the work, we must integrate the force function. That means we need to find k for this particular spring. We will use joules as the unit of work, so we must make sure we use newtons for force and meters for distance.

We are told that $40\ N$ of force is required to hold the spring $10\ cm = 0.1\ m$ from the natural length ($30 - 20$). That means $40 = k \cdot 0.1$, or $k = 400$.

We want to move the spring from $35\ cm$ to $38\ cm$, which is $0.15\ m$ to $0.18\ m$ beyond the natural length. Thus, the work is:

$$W = \int_{0.15}^{0.18} 400x\,dx = 1.98\ J$$

3
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integral(400*x,x,.15,.18)
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1.9799999999999998
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Example 3

We have a cable that weighs 2 lbs/ft attached to a bucket filled with coal that weighs 800 lbs. How much work (in foot-pounds) does it take to raise the bucket from the bottom of a 500 ft mine shaft?

Solution The force involved is the weight of the cable plus the weight of the bucket. The weight of cable depends on how much cable is left.

Let x be the length of cable remaining. Then the force is $F(x) = 2x + 800$.

Therefore, the work is

$$W = \int_0^{500} 2x + 800\, dx = 650,000 \text{ ft-lb}.$$

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integral(2*x+800,x,0,500)
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650000
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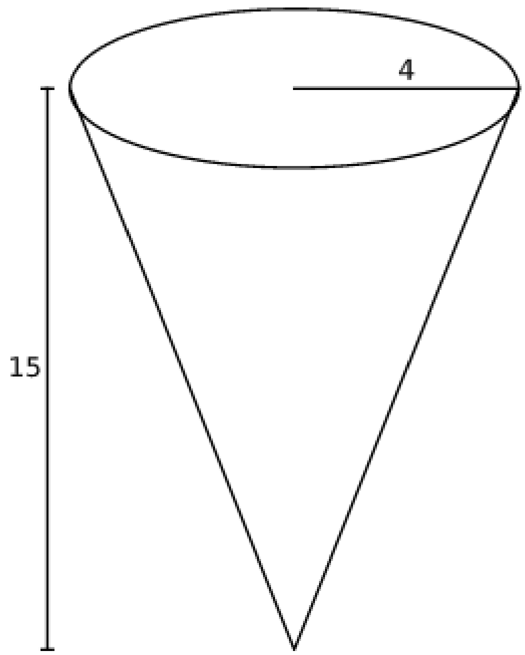
Example 4

A tank has the shape of an inverted circular cone with height 15 ft and base radius 4 ft. It is filled with water to a depth of 12 ft. Find the work needed to pump all the water to the top of the tank (the pump is floating on the water). Note: Water weighs $62.4\ \text{lbs}/\text{ft}^3$.



Solution We can't find a simple force function $F(x)$, so this problem is more complicated than the previous examples.

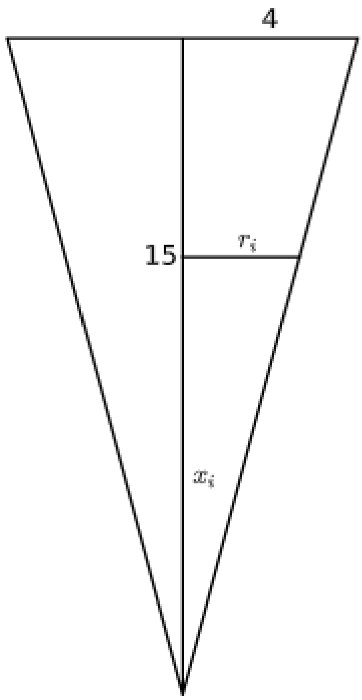
Let x be the vertical distance from the bottom of the tank ($x = 0$). The top of the tank is at $x = 15$, and the water occupies the interval from 0 to 12.

Divide $[0, 12]$ into n equal subintervals of width Δx, and let x_i be a point in the i^{th} subinterval.

For each subinterval we'll approximate the water with a cylinder of radius r_i and height Δx.

By similar triangles, $\frac{r_i}{x_i} = \frac{4}{15}$, so $r_i = \frac{4}{15} x_i$.

Here is a picture of a cross section of the tank:



Force is provided by the weight of the water, which is $62.4\ \mathrm{lbs}/\mathrm{ft}^3$ times the volume (in cubic feet).

The volume of water in the i^{th} subinterval is

$$V_i \approx \text{volume of cylinder} = \pi r^2 h = \pi r_i^2 \Delta x = \pi \left(\frac{4}{15} x_i\right)^2 \Delta x.$$

So the force (= weight of water) for the i^{th} subinterval is

$$F_i = 62.4 V_i \approx 62.4\pi \left(\frac{4}{15} x_i\right)^2 \Delta x = \frac{1664\pi}{375} x_i^2 \Delta x.$$

The distance required to get the water from the i^{th} subinterval to the top of the tank is $d_i \approx 15 - x_i$, so the work required to pump the i^{th} subinterval is

$$W_i = F_i d_i \approx \frac{1664\pi}{375} x_i^2 \Delta x (15 - x_i).$$

The total work reqired is $W \approx \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} \frac{1664\pi}{375} x_i^2 (15 - x_i) \Delta x$.

To get the actual value of the work, we take the limit as $n \to \infty$. Notice that the sum above is a Riemann sum, so

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1664\pi}{375} x_i^2 (15 - x_i) \Delta x = \int_0^{12} \frac{1664\pi}{375} x^2 (15 - x)\, dx \approx 48,178 \text{ ft-lb.}$$

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integral(1664*pi/375*x^2*(15-x),x,0,12)
N(_)
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1916928/125*pi
48177.6553780846
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